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Shear layer instability of an inviscid compressible fluid

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The stability of parallel shear flow of an inviscid compressible fluid is investigated by a linear analysis. The extension of the Rayleigh stability criterion and Howard's semi-circle theorem to compressible flows, obtained by Lees & Lin (1946) and Eckart (1963) respectively, are each rederived by a different approach. It is then shown that a subsonic neutral solution of the stability equation may be found when the basic flow is represented by the hyperbolic-tangent velocity profile. With the aid of this solution, the unstable eigenvalues, eigenfunctions and Reynolds stress are determined by numerical methods. A brief discussion of the results follows.

1. Introduction

The stability of parallel shear flow of an inviscid, homogeneous fluid to infinitesimal two-dimensional non-divergent disturbances has pervaded the scientific literature for the past century. Although this model is somewhat limited in application, the mathematical techniques and physical insight into the mechanism of inertial instability provided by this problem have proved invaluable to studies of more realistic problems. An interesting extension of this classical homogeneous fluid model is the linear stability of parallel shear flow of a compressible perfect gas (e.g. see Betchov & Criminale 1967, for a brief review). The Mach number appears as a parameter in this problem and the results established for homogeneous fluids are recovered as the sound speed $a^* \rightarrow \infty$, i.e. the fluid tends toward incompressibility. In investigations of the compressible flow problem, the pressure field of the basic steady state is usually assumed constant throughout the fluid medium. Then the shallow water equations (e.g. Courant & Friedrichs 1948) and the equations governing motions in an infinite barotropic atmosphere (Obukhov 1949) are analogues of the perturbation equations for compressible flow. In these cases the Froude number replaces the Mach number as the relevant parameter.

Haurwitz (1931) apparently first derived the stability equation for basic flows with a continuous distribution of velocity and temperature. However he did not attempt to determine stability characteristics with continuous basic flows. Until recently the stability of a compressible fluid has mainly been studied as a Kelvin–Helmholtz problem in an infinite medium. This work has been reviewed briefly by Gerwin (1968). Recent studies with continuous profiles have been reported by Lees & Reshotko (1962) and Mack (1965). Due to the mathematical complexities of this latter problem, general stability criteria have not been found as readily as in the homogeneous fluid model. In fact an analytical neutral eigensolution, corresponding to some particular basic velocity and temperature profile, apparently has not been found. For this reason, the unstable eigenvalues are generally found by a careful search procedure (e.g. Mack 1965) which could use a large amount of computer time in delineating the region of instability.

In order to simplify the stability problem, the present investigation will focus on the stability of a basic parallel shear flow to two-dimensional disturbances in a perfect gas, whose basic thermodynamic state is constant. With this simplification, unstable eigenvalues for the hyperbolic-tangent velocity profile are readily found by numerical integration, with the aid of an analytical neutral wave solution. These results also apply to the wider class of fluid systems, indicated above, which are analogues of this compressible fluid model.

2. Basic equations

We shall consider the linear stability of the basic plane parallel flow, $\bar{u}^*(y^*)$, of an ideal gas moving in the x^* direction, with transverse variations along the y^* axis. The basic thermodynamic state is constant and is characterized by the sound speed

$$a^* \equiv \gamma\bar{p}^*/\bar{\rho}^*, \tag{1}$$

where $\bar{p}^*$ and $\bar{\rho}^*$ are pressure and density respectively and γ denotes the ratio of specific heats. Superposed on this basic state are small disturbances in the (x^*, y^*) components of velocity, (u^*, v^*), and pressure p^*.

Non-dimensionalization will be carried out by introducing a velocity scale U and length scale L, which are characteristic of the transverse variations of the basic current $\bar{u}^*(y^*)$. We then define the dimensionless co-ordinates, time, velocities and pressure as

$$\left.\begin{gathered}(x,y) \equiv (x^*,y^*)/L, \quad t \equiv t^*U/L, \\ \bar{u} \equiv \bar{u}^*/U, \quad (u,v) \equiv (u^*,v^*)/U, \\ \pi \equiv p^*/\bar{\rho}^*U^2.\end{gathered}\right\} \tag{2}$$

The Mach number is $M \equiv U/a^*$. Then the basic system of inviscid equations becomes (e.g. Betchov & Criminale 1967)

$$u_t + \bar{u}u_x + v\bar{u}_y = -\pi_x, \tag{3}$$

$$v_t + \bar{u}v_x = -\pi_y, \tag{4}$$

$$M^2(\pi_t + \bar{u}\pi_x) + u_x + v_y = 0, \tag{5}$$

where (3) and (4) are the momentum equations, and the equations of mass continuity and entropy conservation have been combined to yield (5).

Each wave disturbance will be represented in the form

$$q = \hat{q}(y)\exp[i\alpha(x-ct)], \tag{6}$$

where q is u, v or π, α is the real x wave-number and $c = c_r + ic_i$ is the complex phase velocity. The stability problem is to determine the complex eigenvalues c under the conditions that each wave disturbance (6) satisfies the linear equations and boundary conditions separately. Instability corresponds to $c_i > 0$ and consequently an exponential growth of the wave disturbance at the rate αc_i.

We shall make use of the differential equations for the amplitudes of pressure $\hat{\pi}$ and transverse velocity component $\hat{v}$. These equations may be obtained from (3), (4), (5) and (6) in the form

$$(\bar{u}-c)\,\hat{\pi}'' - 2\bar{u}'\hat{\pi}' - \alpha^2(\bar{u}-c)\,[1-M^2(\bar{u}-c)^2]\,\hat{\pi} = 0 \tag{7}$$

and

$$([(\bar{u}-c)\,\hat{v}' - \bar{u}'\hat{v}]/[1-M^2(\bar{u}-c)^2])' - \alpha^2(\bar{u}-c)\,\hat{v} = 0, \tag{8}$$

where a prime denotes differentiation with respect to y.

According to Lees & Lin (1946), there are three types of disturbances associated with (7) or (8). These are classified as subsonic, sonic, or supersonic depending on whether the relative phase velocity $c-\bar{u}$ is less than, equal to or greater than the sound speed (1), i.e. $|c-\bar{u}| \lesseqgtr M^{-1}$. The subsonic disturbances are the counterpart of the so-called inertial modes, which are solutions of (8) in the limiting case $M = 0$ (the Rayleigh stability equation). The physical significance of the sonic disturbances is apparently not clear and will not be considered further. Finally, supersonic disturbances correspond to compression or sound waves. Stable modes of this type, moving in the positive or negative x direction, satisfy

$$c_r > \bar{u}_{\text{max}} + M^{-1},$$
$$c_r < \bar{u}_{\text{min}} - M^{-1}, \tag{9}$$

where $\bar{u}_{\text{max}}$ and $\bar{u}_{\text{min}}$ denote the maximum and minimum values of $\bar{u}$ respectively. In the shallow water and barotropic models, these latter modes are simply gravity waves.

In the present investigation we shall consider the stability of parallel shear flow to two-dimensional subsonic disturbances. Accordingly, if the fluid is unbounded the solution of (7) approaches

$$\hat{\pi} \sim \exp(\mp\alpha[1-M^2(\bar{u}-c)^2]^{\frac{1}{2}}\,y) \quad \text{as} \quad y \to \pm\infty, \tag{10}$$

where it has been assumed that $\bar{u}$ approaches a constant value. Then, the boundary conditions become

$$\hat{\pi} = \hat{v} = 0 \quad (y = \pm\infty). \tag{11}$$

On rigid boundaries the normal velocity must vanish. Thus, from (4),

$$\hat{v} = \hat{\pi}' = 0 \quad (y = y_1, y_2). \tag{12}$$

3. Stability characteristics

Some general stability characteristics have already been derived by Lees & Lin (1946) and later summarized by Lin (1953). They established that, as in the Rayleigh stability problem, a necessary and sufficient condition for the existence of a neutrally stable subsonic disturbance is

$$\bar{u}'' = 0 \quad \text{at} \quad y = y_s, \tag{13}$$

where y_s is the point where $\bar{u} = c_r$. Here we shall show that Fjørtoft's (1950) extension of Rayleigh's theorem is also a sufficient condition for stability to subsonic disturbances. The method of proof is due to Drazin & Howard (1966).

The perturbation energy equation, derived from (3), (4) and (5), is

$$\frac{\partial}{\partial t}\iint \mathscr{E}\,dx\,dy = -\iint uv\overline{u}'\,dx\,dy, \tag{14}$$

where
$$\mathscr{E} \equiv \tfrac{1}{2}(u^2+v^2+M^2\pi^2) \tag{15}$$

denotes the sum of disturbance kinetic plus elastic energy per unit mass and the right-hand side of (14) is the rate of energy conversion by the Reynolds stress. In the case of a homogeneous fluid, the elastic energy is replaced by the available potential energy due to free surface displacements, while in a barotropic atmosphere the available potential energy is associated with pressure oscillations at the lower boundary. Cyclic continuity of all variables is assumed in the x direction and either (11) or (12) applies at the boundaries y_1 and y_2.

The equation for the perturbation potential vorticity ω may be written

$$\frac{\partial\omega}{\partial t}+\overline{u}\frac{\partial\omega}{\partial x}-v\overline{u}'' = 0, \tag{16}$$

where
$$\omega \equiv v_x - u_y + M^2\overline{u}'\pi. \tag{17}$$

From (16) we obtain

$$\frac{\partial}{\partial t}\iint \frac{\overline{u}}{\overline{u}''}\frac{\omega^2}{2}\,dx\,dy = \iint v\omega\overline{u}\,dx\,dy. \tag{18}$$

Upon integration by parts and use of (3) and (5), we get

$$\frac{\partial}{\partial t}\iint\left\{\frac{\overline{u}}{\overline{u}''}\frac{\omega^2}{2}+M^2\overline{u}\pi u\right\}dx\,dy = \iint uv\overline{u}'\,dx\,dy. \tag{19}$$

Addition of (14) and (19) yields, after rearrangement,

$$\frac{\partial}{\partial t}\frac{1}{2}\iint\left\{(1-M^2\overline{u}^2)\,u^2+v^2+M^2(\pi+\overline{u}u)^2+\frac{\overline{u}}{\overline{u}''}\omega^2\right\}dx\,dy = 0. \tag{20}$$

If
$$M\overline{u} = \overline{u}^*/a^* < 1,$$
$$\overline{u}/\overline{u}'' > 0, \tag{21}$$

an initial increase in the perturbation kinetic energy remains bounded because

$$\frac{\partial}{\partial t}\frac{1}{2}\iint\left\{M^2(\pi+\overline{u}u)^2+\frac{\overline{u}}{\overline{u}''}\omega^2\right\}dx\,dy$$

must simultaneously decrease. Thus subsonic flow is stable if $\overline{u}/\overline{u}'' > 0$.

Eckart (1963) has extended Howard's semi-circle theorem, restricting the range of unstable eigenvalues, to more general compressible flows than those considered here. This theorem will be rederived for the present model using the normal mode approach. If $c_i \neq 0$ and $\overline{u}$ finite, then (7) may be divided by $(\overline{u}-c)^3$, with the result

$$([\overline{u}-c]^{-2}\hat{\pi}')'-\alpha^2([\overline{u}-c]^{-2}-M^2)\,\hat{\pi} = 0. \tag{22}$$

Multiplication of (14) by $\hat{\pi}^*$, the complex conjugate of $\hat{\pi}$, and application of (11) and/or (12) yields

$$\int_{y_1}^{y_2}([\overline{u}-c]^{-2}\,|\hat{\pi}'|^2+\alpha^2\{[\overline{u}-c]^{-2}-M^2\}\,|\hat{\pi}|^2)\,dy = 0, \tag{23}$$

where the boundaries at y_1 and/or y_2 may be at infinity. This equation (23) has the same form as (3.1) in Howard's (1961) investigation of the stability of parallel shear flow of an incompressible fluid of variable density. Then it follows that

$$0 \geqslant \{[c_r - \tfrac{1}{2}(\bar{u}_{\max} + \bar{u}_{\min})]^2 + c_i^2 - [\tfrac{1}{2}(\bar{u}_{\max} - \bar{u}_{\min})]^2\} \int_{y_1}^{y_2} Q\, dy + (\alpha M)^2 \int_{y_1}^{y_2} |\hat{\pi}|^2\, dy, \tag{24}$$

where

$$Q \equiv |\bar{u} - c|^{-4} (|\hat{\pi}'|^2 + \alpha^2 |\hat{\pi}|^2). \tag{25}$$

Since $(\alpha M)^2 \geqslant 0$ and $Q > 0$, then (24) implies

$$[c_r - \tfrac{1}{2}(\bar{u}_{\max} + \bar{u}_{\min})]^2 + c_i^2 \leqslant [\tfrac{1}{2}(\bar{u}_{\max} - \bar{u}_{\min})]^2. \tag{26}$$

We note from (24) that increasing values of αM plays the same role in reducing the allowable range of unstable eigenvalues as increasing values of the Richardson number, in Howard's result.

4. A neutral subsonic disturbance

The stability characteristics of the hyperbolic-tangent velocity profile $\bar{u} = 0{\cdot}5(1 + \tanh y)$ in an inviscid homogeneous fluid has been studied in some detail by Michalke (1964).

His numerical analysis proceeded from a knowledge of a neutrally stable eigenfunction solution. This approach will be adapted to the present problem using the profile

$$\bar{u} = \tanh y, \quad -\infty \leqslant y \leqslant \infty. \tag{27}$$

We first note that (7) and (8) reduce to the stability equations for parallel flow in a homogeneous fluid, when $M = 0$. In this case the neutral wave solution of (7) may be obtained from Garcia's (1956) solution for the stream function, and is given by

$$\hat{\pi} = \operatorname{sech} y, \quad c_r = \bar{u}(0) = 0, \tag{28}$$

where the only neutral eigenvalue is $\alpha = 1$. Guided by this result, we shall try as a solution of (7)

$$\hat{\pi} = A \operatorname{sech}^{\beta} y, \tag{29}$$

where A and β are constants to be determined. Substitution of (27) and (29) into (7) leads to a solution if

$$\left.\begin{aligned} c_r = \bar{u}(0) = 0, \\ \beta = \alpha^2, \\ \alpha^2 + M^2 = 1. \end{aligned}\right\} \tag{30}$$

Since (13) is satisfied by the hyperbolic-tangent profile (27) and phase speed given by (30), then (29) is a neutrally stable solution of (7). The locus of neutral eigenvalues, given by (30), is a circle of unit radius in the α, M plane.

The corresponding solutions for $\hat{v}$ and $\hat{u}$, found from (4), (5) and (6) are

$$\hat{v} = -i\alpha A (\operatorname{sech} y)^{1-M^2} \tag{31}$$

and

$$\hat{u} = -A \tanh y (\operatorname{sech} y)^{1-M^2}. \tag{32}$$

The value of A will be discussed in conjunction with the determination of the unstable eigenfunctions in the following section.

5. Numerical computations of the unstable disturbances

Eigenvalues and growth rates

The numerical approach used by Michalke (1964) is well adapted for use in the present problem. In this method the numerical computations of the unstable eigenvalues are simplified by introduction of the transformations

$$\hat{\pi}'/\hat{\pi} = \Pi(y) = \Pi_r + i\Pi_i \tag{33}$$

and

$$z = \tanh y. \tag{34}$$

The latter transformation reduces the integration range to

$$-1 \leqslant z \leqslant 1. \tag{35}$$

After (27), (33) and (34) are introduced into (7), and the real and imaginary parts separated, a coupled system of first-order equations for the variables Π_r and Π_i

M	α	c_i	αc_i
0·0	0·445	0·427	0·190
0·1	0·433	0·433	0·187
0·2	0·426	0·426	0·181
0·3	0·417	0·411	0·171
0·4	0·409	0·386	0·158
0·5	0·397	0·356	0·141
0·6	0·370	0·330	0·122
0·7	0·326	0·309	0·101
0·8	0·279	0·279	0·078
0·9	0·208	0·264	0·055
1·0	0·000	0·000	0·000

TABLE 1

is obtained. These equations, together with the derivation of the boundary conditions, may be found in the appendix. Computations of c_i and αc_i were made over the region $0 \leqslant \alpha^2 + M^2 < 1$ at intervals $\Delta\alpha = \Delta M = 10^{-1}$, with $c_r = 0$. However, in some regions smaller intervals were used, in order to delineate rapid changes in c_i. Isolines of constant growth rate αc_i in the α, M plane, are shown in figure 1. The maximum value of αc_i occurs on the α axis at $\alpha = 0{\cdot}4446$, as found by Michalke (1964).† The dashed line is a curve of maximum αc_i for $M > 0$. The numerical values along this curve are given in table 1.

The functions Π_r and Π_i along the maximum growth rate curve are displayed in figure 2. Π_r and Π_i are antisymmetric and symmetric functions of y respectively. The amplitude of $\Pi_r \to 0$ as $M \to 1, \alpha \to 0$; while $\Pi_i\,(z = \pm 1)$ reaches its maximum amplitude when $M \simeq 0{\cdot}9$, $\alpha \simeq 0{\cdot}208$, then diminishes to zero as $M \to 1$, $\alpha \to 0$.

When the shear flow is antisymmetric (27), Howard (1963) has pointed out that the symmetry in the direction of wave propagation may be preserved by

† The computed eigenvalues c_i, and therefore the growth rates, along the α axis are twice as large as those computed by Michalke, because the amplitude of his basic velocity profile is one-half the value used in (27).

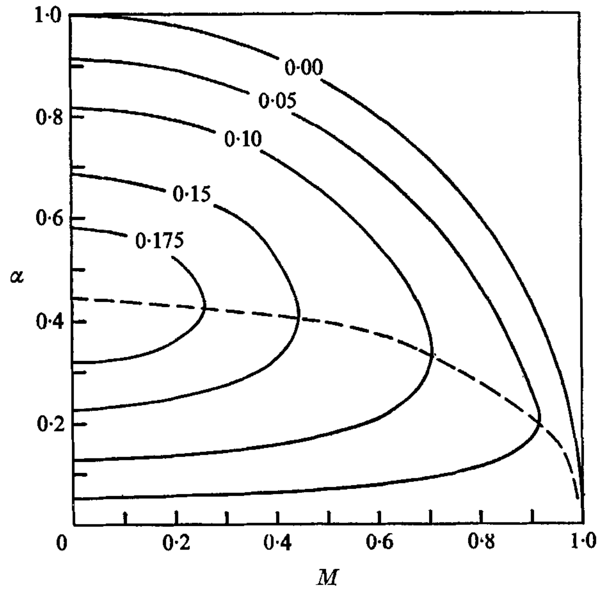

FIGURE 1. Isolines of growth rate αc_i in the α, M plane. The maximum growth rate for a given M occurs along the dashed line.

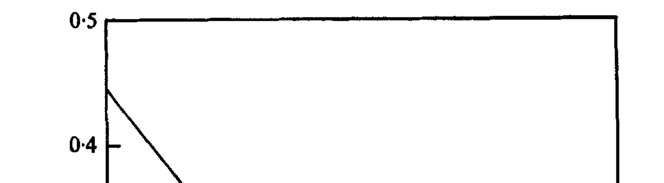

FIGURE 2. The functions Π_r and Π_i, defined by (33), for various values of M along the maximum growth rate curve.

two waves moving in equal and opposite directions, as well as by a stationary wave ($c_r = 0$). Here we have shown that the locus of stable eigenvalues (30) forms a neutral curve which is adjacent to unstable waves with $c_r = 0$. However, the possibility that unstable waves could also exist with $c_r \neq 0$ has not been investigated.

Eigenfunctions and Reynolds stress

The eigenfunctions, $\hat{\pi}_r$ and $\hat{\pi}_i$, may be determined from (7), using the computed eigenvalues. In conformity with Michalke (1964), the integration may be carried out in $0 \leqslant y < \infty$ since $\hat{\pi}_r$ and $\hat{\pi}_i$ are symmetric and antisymmetric functions of y respectively. This follows from (33), using the fact that Π_r and Π_i are antisymmetric and symmetric functions of y.

The constant A in (29) is still undetermined. We shall fix A by requiring that, when $M = 0$ and $\alpha = 1$, $A = 1$ and that the eigenfunctions at $\alpha = 0$, $M = 1$ be equal to the neutral eigenfunctions. A value which accomplishes this is $A = \alpha^2$. Accordingly we take as starting values for the integration

$$\hat{\pi}_r(0) = \alpha^2, \quad \hat{\pi}_i(0) = 0. \tag{36}$$

We obtain from (23) and (36).

$$\hat{\pi}_r'(0) = 0, \quad \hat{\pi}_i'(0) = \alpha^2 \Pi_i(0). \tag{37}$$

When $M = 0$ and $\alpha = 1$, (29) and (30) yield

$$\hat{\pi} = \operatorname{sech} y, \tag{38}$$

which agrees with (36) and (37) when $\alpha = 1$, since $\Pi_i(y) = 0$. Initial conditions (36) and (37) also ensure that $\hat{\pi}(y) = 0$ when $\alpha = 0$ ($0 \leqslant M < 1$), and the neutral solution (29) likewise vanishes at $\alpha = 0$. Michalke (1964) normalized his initial values by choosing the arbitrary constant equal to unity. This was an unfortunate choice, since the solution for the stream function amplitude for $\alpha = 0$ does not vanish at $y = \pm\infty$ as required by his boundary condition (4). However, the computation of the eigenvalues is unaffected and the pressure and velocity components do vanish at the boundaries.

Some eigenfunctions $\hat{\pi}$, $\hat{v}$ and $\hat{u}$ along the neutral curve $\alpha^2 + M^2 = 1$, given by (29), (31) and (32) are shown in figure 3; while in figure 4 some unstable eigenfunctions $\hat{\pi}$, computed along the maximum growth rate curve, are displayed.

The initial value of the Reynolds stress, averaged over one wavelength, may be expressed as

$$\tau = -\overline{\operatorname{Re} u \operatorname{Re} v} = -\tfrac{1}{2}(\hat{u}_r \hat{v}_r + \hat{u}_i \hat{v}_i). \tag{39}$$

The eigenfunctions $\hat{u}$ and $\hat{v}$, needed to calculate τ, may be determined from (3) and (4). These functions are given by

$$\hat{v} = i\alpha^{-1}(\overline{u} - c^*)\,\hat{\pi}'/|\overline{u} - c|^2, \tag{40}$$

$$\hat{u} = -(\overline{u} - c^*)\,(\hat{\pi} - i\alpha^{-1}\overline{u}'\hat{v})/|\overline{u} - c|^2, \tag{41}$$

where c^* is the complex conjugate of c.

The distribution of τ, which is a symmetric function of y, is shown in figure 5. Throughout the region of instability, along the maximum growth rate curve, $\tau \geqslant 0$ for all y, except in a few cases when $\tau < 0$ ($y > 2{\cdot}2$). In each of these cases $|\tau|$ is three orders of magnitude less than $\tau(0)$. However, these values fall at the

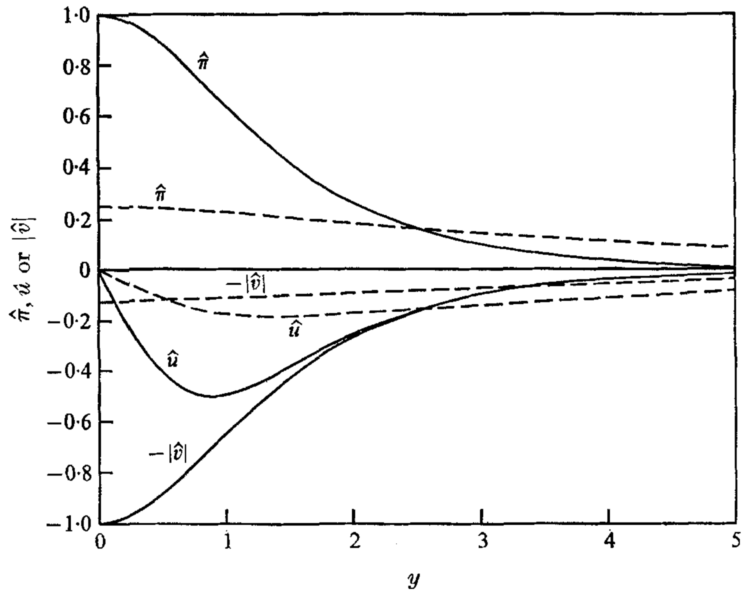

FIGURE 3. Neutral eigenfunctions $\hat{\pi}$, $\hat{u}$ and $|\hat{v}|$ as functions of y. The solid and dashed curves correspond to $M = 1{\cdot}0$ and $M = 0{\cdot}5$ respectively.

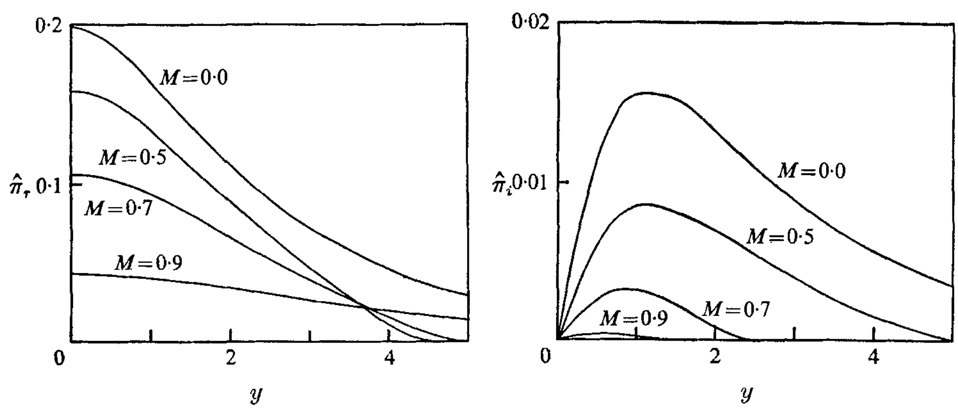

FIGURE 4. The real and imaginary parts of the unstable eigenfunction $\hat{\pi}$ for various values of M along the maximum growth rate curve.

limit of accuracy of the numerical computations and may be spurious. The cusp in τ at the origin sets in at approximately $M = 0{\cdot}10$ along the maximum growth rate curve.

Since $\bar{u}' = \operatorname{sech}^2 y > 0$, the Reynolds stress acts to convert the kinetic energy of the basic horizontal shear flow into disturbance energy. A possible implication of the distribution of $d\tau/dy$ is discussed below. Finally, it may be verified by direct computation that the neutral waves, whose amplitudes are given by (31) and (32), produce $\tau(y) = 0$ as expected from the analysis of Lees & Lin (1946).

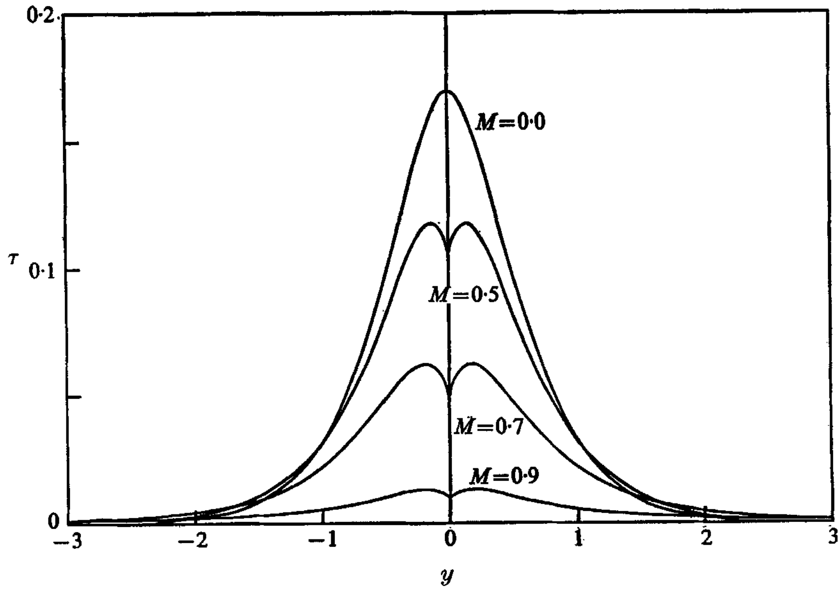

FIGURE 5. The Reynolds stress distribution for various values of M along the maximum growth rate curve.

6. Discussion

The results of this investigation differ from those of the homogeneous non-divergent fluid model because compressible fluid motions are permitted (or equivalently, free surface displacements of a homogeneous fluid). This extra degree of freedom is a stabilizing feature, which is evidenced by a diminution of the unstable region in the α, M plane and a decrease in the growth rate as M increases. Physically, this stabilizing effect arises because a certain amount of basic flow energy must be used to do work against the force due to the elasticity of the medium, before it becomes available to initiate instability. Investigations of the stability of parallel flow of a density stratified fluid under the action of gravity show a similar stabilizing feature, because some of the available basic flow energy must be used to do work against the buoyancy force. For example, Drazin (1958) showed that an increase in the Richardson number, $J > 0$, decreases the unstable region in the α, J plane.

Another interesting feature is the cusp in the Reynolds stress at $y = 0$.† Qualitatively, the distribution of τ in figure 5 shows a small region, in the vicinity of the origin ($0 \leqslant y \lesssim 0{\cdot}2$), where u momentum is accumulating. On the negative

† A similar feature was found by Gilman (1969), who investigated the stability of a baroclinic flow in a zonal magnetic field. However, in Gilman's study, the cusp appeared in the horizontal and vertical heat transport or thermal stress.

side ($0 \geqslant y \gtrsim -0{\cdot}2$), there is a flux divergence of momentum. The initial second-order effect, produced by this particular distribution of $d\tau/dy$, would be a tendency toward the creation of a discontinuous or Kelvin–Helmholtz shear layer. However, under these circumstances, second-order calculations with an inviscid model (e.g. Lin & Benney 1962) may not be justified. A more detailed description of the jump in $d\tau/dy$ at the critical point would require the consideration of viscous effects.

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Appendix: Basic equations and boundary conditions for the determination of the unstable eigenvalues

Upon introduction of (27), (33) and (34) into (7), we obtain

$$\frac{d\Pi}{dz} = \frac{\alpha^2(1-M^2(z-c)^2)-\Pi^2}{1-z^2} + \frac{2\Pi}{z-c}. \tag{A 1}$$

Separation of the real and imaginary parts of (A 1) yields

$$\left.\begin{aligned} \frac{d\Pi_r}{dz} &= \frac{\alpha^2(1-M^2(z^2-c_i^2))-\Pi_r^2+\Pi_i^2}{1-z^2} + \frac{2}{z^2+c_i^2}(z\Pi_r - c_i\Pi_i), \\ \frac{d\Pi_i}{dz} &= \frac{2(\alpha^2M^2c_iz-\Pi_i\Pi_r)}{1-z^2} + \frac{2}{z^2+c_i^2}(c_i\Pi_r + z\Pi_i). \end{aligned}\right\} \tag{A 2}$$

When $|y|$ is large, the behaviour of $\hat{\pi}(y)$ is given by (10), with

$$\bar{u}(\pm\infty) = (\tanh y)_{y=\pm\infty} = \pm 1.$$

Then it follows that

$$\left.\begin{aligned} \Pi(1) &= -\alpha(1-M^2(1-c)^2)^{\frac{1}{2}}, \\ \Pi(-1) &= \alpha(1-M^2(1+c)^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{A 3}$$

The transformed boundary conditions may be obtained by squaring the expressions in (A 3) and solving for the real and imaginary parts of Π. The result is:

$$\left.\begin{aligned} \Pi_r(1) &= -\alpha 2^{-\frac{1}{2}}([1-M^2(1-c_i^2)] + \{[1-M^2(1-c_i^2)]^2 + (2M^2c_i)^2\}^{\frac{1}{2}})^{\frac{1}{2}}, \\ \Pi_i(1) &= \alpha^2M^2c_i/\Pi_r(1), \\ \Pi_r(-1) &= -\Pi_r(1), \\ \Pi_i(-1) &= \Pi_i(1). \end{aligned}\right\} \tag{A 4}$$

In order to start the numerical integration, boundary values of $d\Pi/dz$ are needed. These may be found from (A 1) by application of L'Hospital's rule, since $d\Pi/dz$ is not determined at $z = \pm 1$. When this evaluation is carried out, we obtain

$$\left.\begin{aligned}
\left.\frac{d\Pi_r}{dz}\right|_{z=1} &= \{(1-\Pi_r)^2+\Pi_i^2\}^{-1}\left(\alpha^2M^2[(1-\Pi_r)+c_i\Pi_i]\right.\\
&\qquad\left.+\frac{2[(\Pi_r-c_i\Pi_i)(1-\Pi_r)-(\Pi_i+c_i\Pi_r)\Pi_i]}{1+c_i^2}\right),\\
\left.\frac{d\Pi_i}{dz}\right|_{z=1} &= \{(1-\Pi_r^2)+\Pi_i^2\}^{-1}\left(\alpha^2M^2[\Pi_i-c_i(1-\Pi_r)]\right.\\
&\qquad\left.+\frac{2[\Pi_i(\Pi_r-c_i\Pi_i)+(1-\Pi_r)(\Pi_i+c_i\Pi_r)]}{1+c_i^2}\right),\\
\left.\frac{d\Pi_r}{dz}\right|_{z=-1} &= \left.\frac{d\Pi_r}{dz}\right|_{z=1},\\
\left.\frac{d\Pi_i}{dz}\right|_{z=-1} &= -\left.\frac{d\Pi_i}{dz}\right|_{z=1}.
\end{aligned}\right\} \qquad (A\,5)$$

Equations (A 2) were integrated from the boundaries at $z = \pm 1$ for fixed $\alpha^2+M^2 < 1$. c_i was chosen by trial and error until the solutions overlapped in $-0{\cdot}1 \leqslant z \leqslant 0{\cdot}1$. These computations were carried out on the CDC 6600 computer, at the National Center for Atmospheric Research, using a Runge–Kutta procedure and an integration step of 0·025, as used by Michalke (1964). However, when $c_i \ll 1$, the integration step was decreased in $-0{\cdot}1 \leqslant z \leqslant 0{\cdot}1$, because the denominators in (A 2) become small. Finally, in order to check on the accuracy of the program, the computations were compared with Michalke's numerical results ($M = 0$) and with the neutral solution (29), (30).

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